

Universality in Sandpile Models

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Abstract

A new classification of sandpile models into universality classes is presented. On the basis of extensive numerical simulations, in which we measure an extended set of exponents, the Manna two state model [S. S. Manna, J. Phys. A 24, L363 (1991)] is found to belong to a universality class of random neighbor models which is distinct from the universality class of the original model of Bak, Tang and Wiesenfeld [P. Bak, C. Tang and K. Wiensfeld, Phys. Rev. Lett. 59, 381 (1987)]. Directed models are found to belong to a universality class which includes the directed model introduced and solved by Dhar and Ramaswamy.

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The introduction of sandpile models as a paradigm of self-organized criticality by Bak, Tang and Wiesenfeld (BTW) [1] stimulated numerous theoretical [2,3] and numerical studies [4–7]. In these models, which are defined on a lattice, grains are deposited randomly until the height at some site exceeds a threshold, and becomes unstable. “Sand” is then distributed to the nearest neighbors. As a result of this relaxation process neighboring sites may become unstable, resulting in a cascade of relaxations called *an avalanche*. It was observed that these models are self-driven into a critical state which is characterized by a set of exponents [1]. These include exponents that describe the distribution of quantities such as avalanche size and lifetime, and exponents which relate these properties of the dynamics. Large scale simulations of the BTW model [4] and some variants of it [8,9] were performed. The BTW model and the Manna two-state model were concluded to belong to the same universality class [8]. Christensen and Olami later introduced an extended set of exponents [7]. They measured the values of these exponents for the BTW model, and gave theoretical predictions and heuristic arguments for the values of some of the exponents. Continuous height models were also studied [10] and some aspects of universality were examined [11]. A sandpile model with a preferred direction was introduced and solved by Dhar and Ramaswamy [3].

In this paper we present simulation results which suggest a new classification of sandpile models into universality classes. The Manna two-state model is found to belong to a universality class of random relaxation models which is distinct from the BTW universality class. We first describe the different models, and define the properties of avalanches, with the exponents characterizing them. The models are defined on a d dimensional lattice of linear size L . Each site is assigned a dynamic variable $E(\mathbf{i})$ which represents some physical quantity such as energy, stress etc. In a critical height model a configuration $\{E(\mathbf{i})\}$ is called *stable* if for all sites $E(\mathbf{i}) < E_c$, where E_c is a threshold value. The evolution between stable configurations is by the following rules:

(i) Adding energy. Given an arbitrary stable configuration $\{E(\mathbf{j})\}$ we select a site \mathbf{i} at random and increase $E(\mathbf{i})$ by some amount δE . When an unstable configuration is reached

rule (ii) is invoked.

(ii) The relaxation rule. If the dynamical variable at site \mathbf{i} exceeds the threshold E_c , relaxation takes place, whereby energy is distributed in the following way:

$$E(\mathbf{i}) \rightarrow E(\mathbf{i}) - \sum_{\mathbf{e}} \Delta E(\mathbf{e}) \quad (1)$$

$$E(\mathbf{i} + \mathbf{e}) \rightarrow E(\mathbf{i} + \mathbf{e}) + \Delta E(\mathbf{e}),$$

where \mathbf{e} are a set of (unit) vectors from the site \mathbf{i} to some neighbors. As a result of the relaxation the dynamic variable in one or more of the neighbors may exceed the threshold. The relaxation rule is then applied until a stable configuration is reached. The sequence of relaxations is an avalanche which propagates through the lattice.

The parameters δE and E_c are irrelevant to the scaling behavior [2,11]. Thus the only factor determining the exponents is the vector ΔE , to be termed *relaxation vector*. For a square lattice with relaxation to nearest neighbors it is of the form $\Delta E = (E_N, E_E, E_S, E_W)$, where E_N for example is the amount transferred to the northern nearest neighbor. The original BTW model is given by the vector $(1, 1, 1, 1)$. The relaxation in the directed model of Dhar and Ramaswamy [3] is specified by any vector with ones in two adjacent directions and zeroes in the two other directions, such as $(0, 0, 1, 1)$. In a random relaxation model a set of neighbors is randomly chosen for relaxation. Such a model is specified by a set of relaxation vectors, each vector being assigned a probability for its application. As an example, a possible realization of a two-state model makes use of the six relaxation vectors $(1, 1, 0, 0), (1, 0, 1, 0), (1, 0, 0, 1), (0, 1, 1, 0), (0, 1, 0, 1)$ and $(0, 0, 1, 1)$, each one applied with a probability of $1/6$. In Manna's two-state model [8] the variable is decreased to zero on relaxation, with sand distributed randomly among the nearest neighbors. We define a current

$$\mathbf{J}[\Delta E] = \sum_{\mathbf{e}} \Delta E(\mathbf{e}) \mathbf{e}, \quad (2)$$

which is the net flow in a relaxation. We also define

$$\mathbf{J} = \sum_{\Delta E} \mathbf{J}[\Delta E] P(\Delta E), \quad (3)$$

which is the current averaged over the ensemble of relaxation vectors. Models can be classified according to the value of the current $\mathbf{J}[\Delta E]$ and its average, \mathbf{J} . A model is called *non-directed* if $\mathbf{J}[\Delta E] = 0$ (the BTW model for example). Random relaxation models such as the Manna two-state model, which satisfy $\mathbf{J}[\Delta E] \neq 0$ and $\mathbf{J} = 0$, are called *non-directed on average*. Models with $\mathbf{J} \neq 0$ are called *directed*. In this paper we present evidence that this is a classification into universality classes.

Avalanches have various properties which can be measured in a simulation: size, area, lifetime, linear size, and perimeter. The size (s) of an avalanche is the total number of relaxation events that occurred in the course of a single avalanche. The area (a) is the number of sites in the lattice where relaxation occurred. Relaxation of all sites which exceed the threshold at a given time is considered a single time step. The lifetime (t) of an avalanche is the number of such steps. As for the linear size of an avalanche, there is no unique choice. A possible choice is the maximal distance (d) between the origin of the avalanche to sites of the avalanche cluster. Another possibility is the radius of gyration (r) of the cluster of sites where relaxation occurred. A site belonging to the cluster of sites visited by an avalanche is defined to be a perimeter site if it has a nearest neighbor where no relaxation took place. The perimeter (p) is the number of perimeter sites. Thus we have a set of variables $\{s, a, t, r, d, p\}$ which characterize an avalanche. The avalanche variables have probability functions which are assumed to fall off with a power law defined by $P(x) \sim x^{1-\tau_x}$, where $x \in \{s, a, t, r, d, p\}$. These variables also scale against each other in the form

$$y \sim x^{\gamma_{yx}}, \quad (4)$$

for $x, y \in \{s, a, t, r, d, p\}$. The exact definition of the γ 's is in terms of conditional expectations values: $E[y|x] \sim x^{\gamma_{yx}}$ [6]. The exponents are not independent. Scaling relations are found in [7]. We just note that

$$\gamma_{yx} = \gamma_{xy}^{-1}, \quad (5)$$

$$\gamma_{zx} = \gamma_{zy}\gamma_{yx}.$$

Avalanches are proven to be compact for BTW type models [7] but have a fractal boundary. It is reasonable to assume that the fractal dimension D_f of the boundary is given by the scaling of the perimeter (p) against the linear size of the avalanche. It seems that for models which are non-directed the radius of gyration is the proper measure of size [11]. Therefore we identify D_f with γ_{pr} . For directed models the maximum distance from the origin to the perimeter is the proper measure of size, and D_f is identified with γ_{pd} . It is accepted that the dynamical exponent z of non-directed models should be identified with γ_{tr} [11]. In the case of directed models we identify the dynamical exponent with γ_{td} .

Having defined the models, we now describe the simulations. We used open boundary conditions and system sizes up to 512^2 , with 5 million grains dropped, in two dimensions; in three dimensions system sizes were up to 112^3 , with 20 million grains dropped. An algorithm due to Grassberger [5] was used. We ascertained the dynamics has reached the critical state by applying Dhar's "burning algorithm" [2], or by starting with a configuration belonging to the critical state. Manna's and our own simulation results for the BTW model indicate that the distribution exponents are system size dependent, with a logarithmic convergence to the infinite system values. The values of the γ 's on the other hand, seem to be almost independent of system size. Moreover, we found that the relations that specify the γ 's hold during avalanches as well, and are not just a scaling property of completed avalanches. Thus the γ 's provide a robust characterization of the dynamical properties of a sandpile model, and can be used for a reliable classification of sandpile models into universality classes.

Previous studies clearly show that directed and non-directed models belong to different universality classes [3,7,8]. On the basis of Manna's simulation results it was concluded that the Manna two-state model and the BTW model are in the same universality class [8]. This conclusion is based on measurements of a limited set of exponents: τ_s, τ_t and γ_{ts} . We measured the extended set of exponents introduced by Christensen and Olami, and the fractal dimension. The γ 's we obtained in two dimensions are listed in Table I. Our results are consistent with known analytical results and simulation data: Dhar and Ramaswamy's analytical solution of a directed model [3]; simulation results and scaling arguments given by

Christensen and Olami [7]; simulation results of Manna [4,8]. A momentum-space analysis of a Langevin equation indicates that for the BTW model $z = (2 + d)/3$ [11]. Our results for γ_{rt} which is identified with $1/z$, confirm this scaling relation. This agreement supports our observation that the γ 's are size independent, and indicates that we are in the right avalanche size regime for the observation of γ_{rt} . On the basis of the difference in the γ 's for the BTW and two-state models we conclude that the two models are not in the same universality class (Fig. 1).

In order to establish that the classification introduced above is a classification into universality classes we provide evidence that some details of the models are irrelevant (Fig. 2). Simulation results of the BTW model on the triangular lattice and square lattice were compared [4,11]. No significant difference was reported. We define N as the number of states of the $E(\mathbf{i})$ in stable configurations of discrete models. When the components of the relaxation vector are all 1's, N also equals the number of neighbors. In sandpile models the question of the lattice dependence or interaction range dependence of the exponents is actually a question of the dependence on N . We observed a crossover effect when increasing N . The scaling obtained for the BTW model on a square lattice ($N=4$) is shifted to larger avalanches when N is increased. Similar cross-over was observed in the other universality classes. Note that the requirement that $\mathbf{J}[\Delta E] = 0$ does not imply isotropy. This is the reason the universality class was called *non-directed*, rather than isotropic. As an example, a model with a toppling vector (1,2,1,2) fulfills this requirement, and simulations show that it belongs to the universality class of non-directed models.

Continuous models were simulated as well. There are two types of realizations of continuous models. In one, the variables are turned into continuous variables, and when the amount of sand added is not a multiple of the amount distributed on relaxation (or is a random variable taking such values) then the height profile is turned into a continuous distribution. The other is the Zhang realization, where on relaxation the dynamic variable is decreased to zero and sand distributed equally among the nearest-neighbors [10]. Both types seem to be in the same universality class [11]. This is indicated by our simulations as

well.

There is a number of possible realizations of a two-state model. The neighbors to which sand is distributed can be chosen as distinct (no neighbor chosen twice) or not. In Manna's two-state model [8] the variable is decreased to zero on relaxation, with sand distributed randomly among the nearest neighbors. In this case the relaxation process depends on the variable value. Continuous variants of the model may also be defined. We have simulated realizations of such models and all were in the same universality class (Fig. 2). Simulations of two-state models were performed with annealed randomness only [12].

On the basis of the wave structure of avalanches in the BTW model [13], it can be shown that avalanches have a “shell” structure, i.e. the sites which relaxed at least $n+1$ times form a connected cluster with no holes which is contained in the cluster of sites which relaxed at least n times (Fig. 3(a)). Avalanches in random relaxation models do not share this property, and their structure is more irregular. A typical avalanche in a two-state model is shown in Fig. 3(b). These geometrical differences reflect in the fractal dimension of the boundary, which is greater for the two-state model.

The distinction between the universality classes of non-directed models and models which are non-directed only on average holds in three dimensions as well (Table II). The difference is less marked because the exponents are nearing their mean field values.

Directed models also form a universality class. In addition to the models studied by Dhar and Ramaswamy, where the relaxation vector is of the form $(1,1,0,0)$ or $(1,1,1,0)$, we simulated models with the relaxation vectors $(1,1,1,2)$ and $(1,1,2,2)$. In the latter, multiple relaxations are possible, but it does not reflect in the scaling behavior. We found the same exponent values in all these models. The values we obtained in simulations (Table I) are in agreement with the analytical solution. Directed models with a random relaxation rule show cross-over.

In summary, using extensive numerical simulations we identified three universality classes in sandpile models: (a) non-directed models (BTW model); (b) random relaxation models which are non-directed only on average (Manna two-state model) and (c) directed models

(Dhar and Ramaswamy's directed model). These universality classes correspond to a classification according to the value of the current $\mathbf{J}[\Delta E]$ and its average. Locally non-conservative models also show SOC and may form a different universality class [9].

Recently Pietronero et al. [14] introduced a novel theoretical framework for calculating the exponents of sandpile models, in a manner which immediately reveals their universality. Within their scheme, which is purely phenomenological, the Manna two-state model and the BTW model are found to be in the same universality class. Its failure to distinguish between the two models indicates that some key ingredient is missing from their scheme. We suspect that multiple relaxation is the missing element. Work is now in progress to extend the procedure to include some form of multiple relaxation.

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REFERENCES

- [1] P. Bak, C. Tang and K. Wiesenfeld, *Phys. Rev. Lett.* **59**, 381 (1987); *Phys. Rev. A* **38**, 364 (1988).
- [2] D. Dhar, *Phys. Rev. Lett.* **64**, 1613 (1990).
- [3] D. Dhar and R. Ramaswamy, *Phys. Rev. Lett.* **63**, 1659 (1989).
- [4] S.S. Manna, *J. Stat. Phys.* **59**, 509 (1990); *Physica A* **179**, 249 (1991).
- [5] P. Grassberger and S.S. Manna, *J. Phys.* **51**, 1077 (1990).
- [6] K. Christensen, H.C. Fogedby, and H.J. Jensen, *J. Stat. Phys.* **63**, 653 (1991).
- [7] K. Christensen and Z. Olami, *Phys. Rev. E* **48**, 3361 (1993).
- [8] S.S. Manna, *J. Phys. A* **24**, L363 (1991).
- [9] S.S. Manna, L. B. Kiss and J. Kertesz, *J. Stat. Phys.* **61**, 923 (1990).
- [10] Y.C. Zhang, *Phys. Rev. Lett.* **63**, 470 (1989).
- [11] A. Diaz-Guilera, *Phys. Rev. E* **45**, 8551 (1992); *Europhys. Lett.* **26** 177 (1994).
- [12] Simulating the two-state model with quenched randomness is not possible because the topology permits the formation of closed loops of relaxation. These loops have no outlet to the boundary and the result is avalanches which last indefinitely.
- [13] E.V. Ivashkevich, D.V. Ktitarev and V.B. Priezzhev, *Physica A* **209**, 347 (1994).
- [14] A. Vespignani, S. Zapperi and L. Pietronero, *Phys. Rev. Lett.* **72**, 1690 (1994); *Phys. Rev. E* **51**, 1711 (1995).

FIGURES

FIG. 1. Simulation results for the BTW model (circles) and two-state model (squares). $E[r|t]$ (average radius of gyration for given avalanche lifetime) vs. t is displayed in (a), yielding γ_{rt} . In (b) we show a graph of $E[s|a]$ vs. a which yields γ_{sa} . Their values are listed in Table I. Data was binned with bin size increasing exponentially. System size is 512^2 , with 10^7 grains dropped. These results indicate that the two models belong to different universality classes.

FIG. 2. Simulation results showing the universality of models in the BTW (non-directed) and two-state (non-directed on average) universality classes. Graph shows $E[s|a]$ vs. a for system size 128^2 . Unless otherwise stated simulations were performed on a square lattice. For models in the BTW class we see data collapse on a curve with $\gamma_{sa} = 1.06$. For the random relaxation models $\gamma_{sa} = 1.23 \pm 0.01$.

FIG. 3. Typical avalanche structure for the BTW model (a) and two-state model (b). Grey-scale indicates the number of relaxations which occurred at each site during an avalanche. White represents zero relaxations, and black represents the maximal no. of relaxations (10 in (a), 45 in (b)). System size is 150^2 . Note the shell structure in the BTW avalanche (an analytically provable property) vs. the irregular structure of the avalanche in the two-state model. These qualitative geometrical differences translate into quantitative differences in exponent value, especially the fractal dimension of the boundary.

TABLES

TABLE I. γ exponents for universality classes in two dimensions. The other γ 's can be found from the scaling relations, Eq. [5]. The values of these exponents were observed to be independent of system size. The typical spread of data for different runs of different models within the universality class is ± 0.01 about the mean.

TABLE II. Exponents in three dimensions for the BTW model and a three-state random relaxation model (non-directed on average). Distribution exponents are given for system size of 96^3 .

Exponent	model		
	BTW	two-state	Directed
$1/z^a$	0.76	0.67	1.00
γ_{st}	1.62	1.70	1.51
γ_{at}	1.53	1.35	1.51
γ_{sa}	1.06	1.23	1.00
D_f	1.26	1.42	1.00

^aIn non-directed models z is identified with γ_{tr} , and in directed models it is identified with γ_{td} .

Exponent	BTW	3-state
τ_s	2.35	2.43
τ_a	2.35	2.46
$\gamma_{rt} (1/z)$	0.60	0.54
γ_{st}	1.78	1.80
γ_{at}	1.78	1.72
γ_{sa}	1.00	1.06

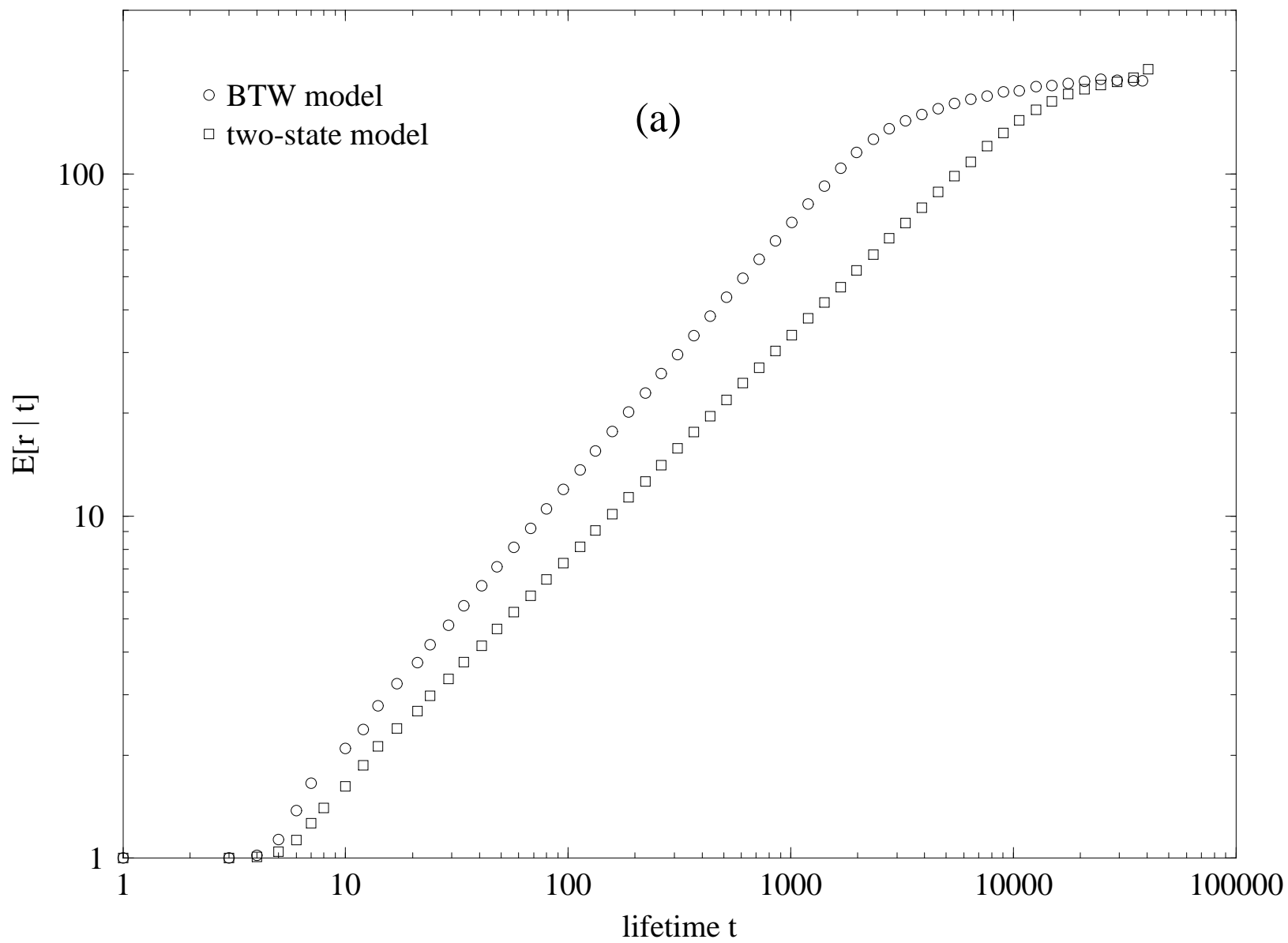


Fig. 1a

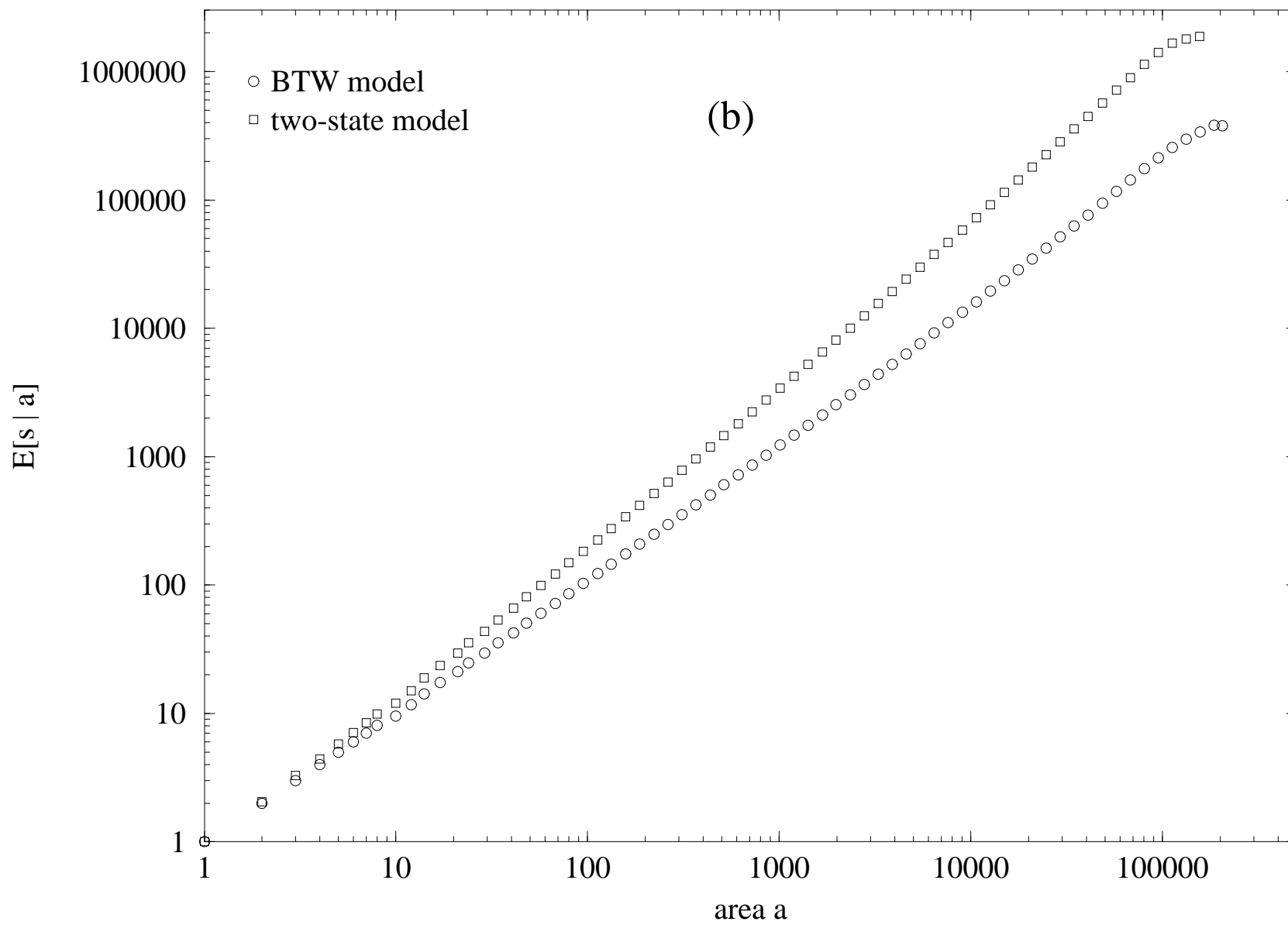


Fig. 1b

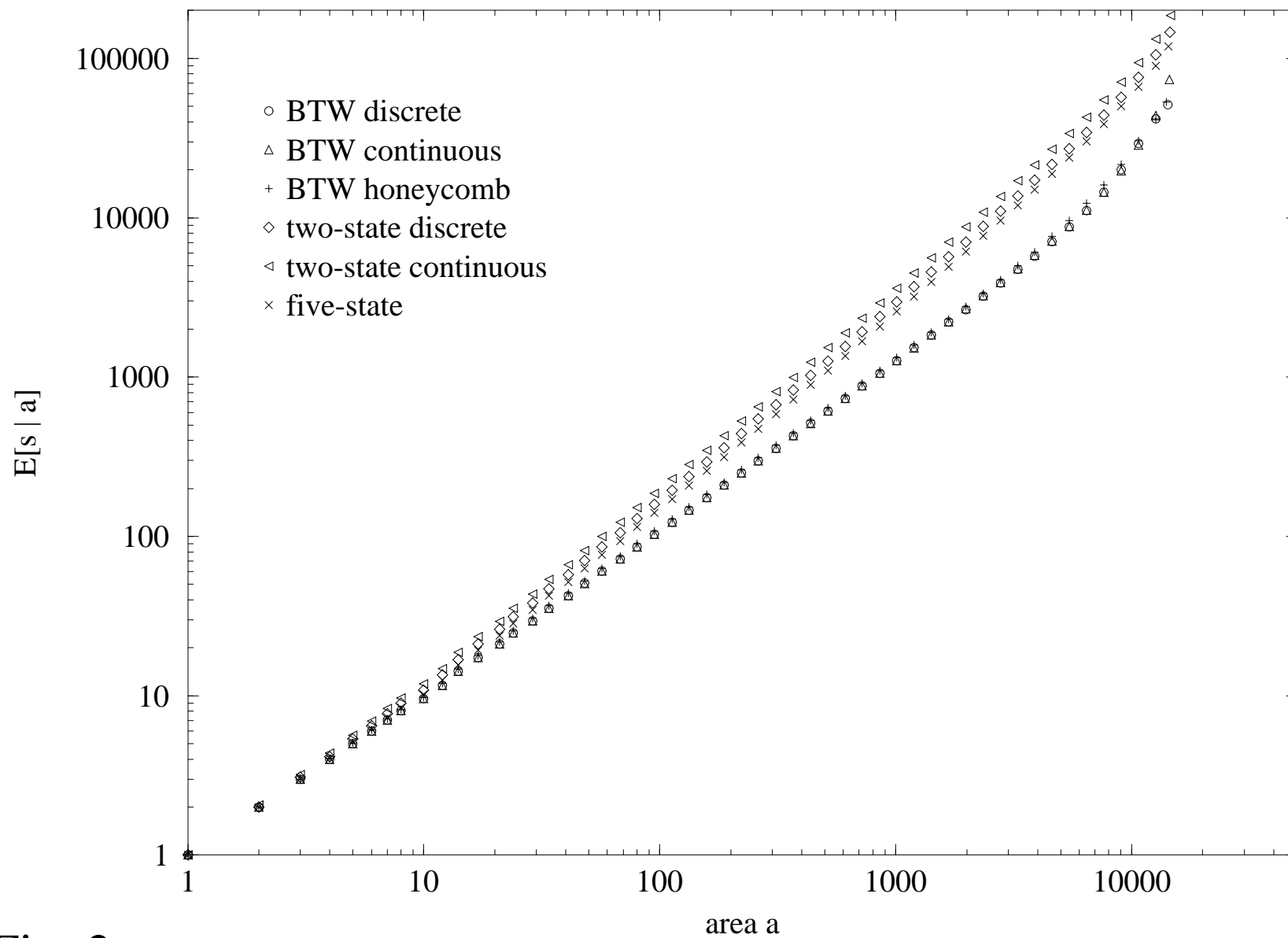


Fig. 2

(a)

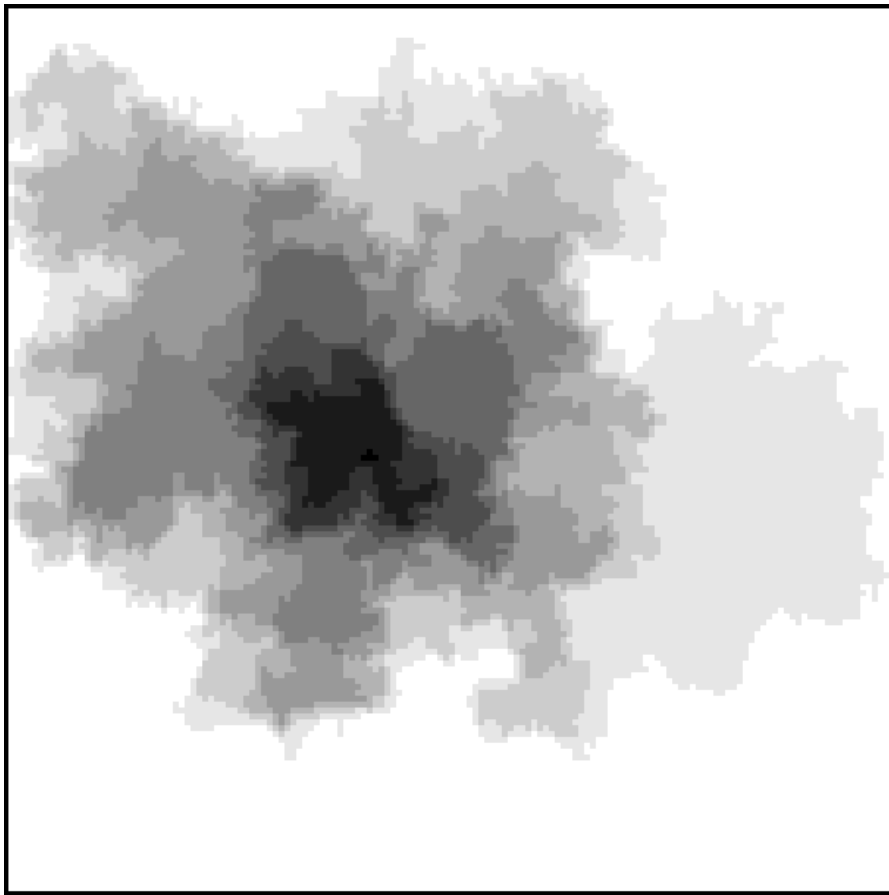


Fig. 3a

(b)

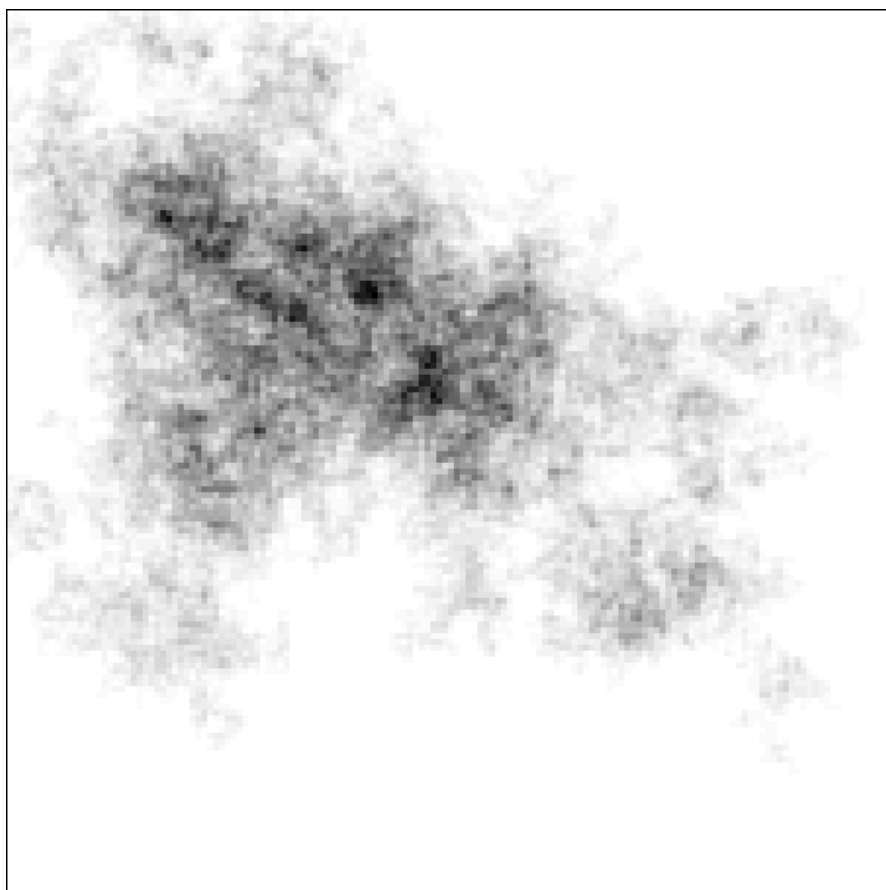


Fig. 3b